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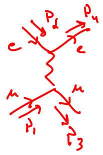
If at some point we had summed over $v\bar{v} \Rightarrow \not{p} - mc$.

Note: There are no spinors left in the expression! Only p_μ's and γs!

$$\boxed{\text{To impose sum over } s_a, s_b \text{ replace } \bar{u}(a)\Gamma_1 u(b)[\bar{u}(a)\Gamma_2 u(b)]^* \Rightarrow \mathrm{Tr}[\Gamma_1(\not{p}_b + m_b c)\bar{\Gamma}_2(\not{p}_a + m_a c)]}$$

Let's put this result to work:

$e + \mu \rightarrow e + \mu$



$$M = -\frac{g_e^2}{(p_4-p_2)^2}\bar{u}(3)\gamma^\mu u(1)\bar{u}(4)\gamma^\nu u(2) g_{\mu\nu}$$

$$|M|^2 = \frac{g_e^4}{(p_4-p_2)^4}\bar{u}(3)\gamma^\mu u(1)\bar{u}(4)\gamma^\nu u(2) g_{\mu\nu}[\bar{u}(3)\gamma^\lambda u(1)\bar{u}(4)\gamma^\alpha u(2) g_{\lambda\alpha}]^*$$

$$\mathrm{Tr}[\gamma^\mu(\not{p}_1 + m_1 c)\bar{\gamma}^\lambda(\not{p}_3 + m_3 c)]$$

$$\mathrm{Tr}[\gamma^\nu(\not{p}_2 + m_2 c)\bar{\gamma}^\alpha(\not{p}_4 + m_4 c)]$$

2 incoming particles
w/ 2 spin states each

$$\langle |M|^2 \rangle = \frac{1}{4}\frac{g_e^4}{(p_4-p_2)^4}\mathrm{Tr}[\gamma^\mu(\not{p}_1 + m_1 c)\bar{\gamma}^\lambda(\not{p}_3 + m_3 c)]\mathrm{Tr}[\gamma^\nu(\not{p}_2 + m_2 c)\bar{\gamma}^\alpha(\not{p}_4 + m_4 c)] g_{\mu\nu} g_{\lambda\alpha}$$

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To continue we need to know how to evaluate traces in spin space. Fortunately there are some useful results:

a) $\mathrm{Tr}(\gamma^\mu \gamma^\alpha \gamma^\lambda \gamma^\beta) = 4(g^{\mu\alpha} g^{\lambda\beta} - g^{\mu\lambda} g^{\alpha\beta} + g^{\mu\beta} g^{\alpha\lambda})$

b) $\mathrm{Tr}(\gamma^\mu \gamma^\alpha \gamma^\lambda) = 0$

c) $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$

$\gamma^0 \gamma^{\lambda\dagger} \gamma^0 = \gamma^\lambda$ (nontrivial but true!)

Then: $\mathrm{Tr}\left[\gamma^\mu (\not{p}_a + m_a c) \bar{\gamma}^\lambda (\not{p}_b + m_b c)\right] = X$

Using: $\mathrm{Tr}[A+B] = \mathrm{Tr}\,A + \mathrm{Tr}\,B$

We have: $X = \mathrm{Tr}(\gamma^\mu \not{p}_a \gamma^\lambda \not{p}_b) + \mathrm{Tr}(\gamma^\mu \not{p}_a \gamma^\lambda m_b c) + \mathrm{Tr}(\gamma^\mu m_a c \gamma^\lambda \not{p}_b) + \mathrm{Tr}(\gamma^\mu m_a c \gamma^\lambda m_b c)$

Using: $\mathrm{Tr}[\alpha M] = \alpha\, \mathrm{Tr}\, M$

α scalar in space where M is a matrix!

$$X = p_{a\alpha} p_{b\beta} \underbrace{\mathrm{Tr}(\gamma^\mu \gamma^\alpha \gamma^\lambda \gamma^\beta)}_{4(g^{\mu\alpha} g^{\lambda\beta} - g^{\mu\lambda} g^{\alpha\beta} + g^{\mu\beta} g^{\alpha\lambda})} + mc\, p_{a\alpha} \underbrace{\mathrm{Tr}(\gamma^\mu \gamma^\alpha \gamma^\lambda)}_{0} - mc\, p_{b\alpha} \underbrace{\mathrm{Tr}(\gamma^\mu \gamma^\lambda \gamma^\alpha)}_{0} + (m_a c)(m_b c) \underbrace{\mathrm{Tr}(\gamma^\mu \gamma^\lambda)}_{4 g^{\mu\lambda}}$$

$$= 4(p_a^\mu p_b^\lambda - g^{\mu\lambda} p_a \cdot p_b + p_a^\lambda p_b^\mu) + 4 m_a m_b c^2 g^{\mu\lambda}$$

Then for our $e + \mu \to e + \mu$ result:

$$\langle |M|^2 \rangle = \frac{1}{4} \frac{g_e^4}{(p_4 - p_2)^4}\, 16\, (p_1^\mu p_3^\lambda - g^{\mu\lambda} p_1 \cdot p_3 + p_1^\lambda p_3^\mu + m_\mu^2 c^2 g^{\mu\lambda})(p_2^\nu p_4^\alpha - g^{\nu\alpha} p_2 \cdot p_4 + p_2^\alpha p_4^\nu + m_e^2 c^2 g^{\nu\alpha})\, g_{\mu\nu} g_{\lambda\alpha}$$

$$= \frac{8 g_e^4}{(p_4 - p_2)^4} \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - (p_1 \cdot p_3) m_\mu^2 c^2 - (p_2 \cdot p_4)^2 m_e^2 c^2 + 2 m_e^2 m_\mu^2 c^4 \right]$$

Note: Our final expression is in terms of only 4momenta, i.e. E and $\vec{p}$!

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Recall that for 2-body scattering: $\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|M|^2}{(E_1+E_2)^2}\frac{|\vec{p}_f|}{|\vec{p}_i|}$ for the CM-frame

If we then take the approximation $m_\mu \gg m_e$ and assume that $E_2 = E_e \ll m_\mu c^2$ we find:

$$|\vec{p}_f| = |\vec{p}_i| \ , \quad E_1 + E_2 = \underbrace{m_\mu c^2 + E_2}_{\text{CM-frame is essentially the rest frame of } \mu} \simeq m_\mu c^2$$

Then: $\quad \left\langle \frac{d\sigma}{d\Omega} \right\rangle = \left(\frac{\hbar}{8\pi m_\mu c}\right)^2 \langle |M|^2 \rangle$

$|\vec{p}_2| = |\vec{p}_4| \equiv |\vec{p}|$

$p_1 = (m_\mu c, 0) \quad p_2 = (\frac{E}{c}, \vec{p}_2) \quad p_3 = (m_\mu c, \vec{0}) \quad p_4 = (\frac{E}{c}, \vec{p}_4)$

approximate since $m_\mu \gg m_e$

$$\text{Then: } (p_4 - p_2)^2 = (0, \vec{p}_4 - \vec{p}_2)^2 = 0 - (\vec{p}_4 - \vec{p}_2)^2 = -\vec{p}_4^{\,2} - \vec{p}_2^{\,2} + 2\vec{p}_2\cdot\vec{p}_4$$
$$= -2\vec{p}^{\,2}(1-\cos\theta)$$
$$= -4\vec{p}^{\,2}\sin^2\frac{\theta}{2}$$

$$p_2 \cdot p_4 = \left(\frac{E}{c}\right)^2 - \vec{p}_2\cdot\vec{p}_4 = m_e^2c^2 + 2\vec{p}^{\,2}\sin^2\left(\frac{\theta}{2}\right) \longleftarrow \left(\frac{E}{c}\right)^2 - p^2\cos^2\theta = \left(\frac{E}{c}\right)^2 - p^2 + p^2 - p^2\cos^2\theta$$
$$= m_e^2c^2 + p^2(1-\cos\theta)$$
$$= m_e^2c^2 + 2p^2\sin^2\frac{\theta}{2}$$

$$p_1 \cdot p_3 = m_\mu^2 c^2$$
$$(p_1\cdot p_2)(p_3 \cdot p_4) = (p_1\cdot p_4)(p_2\cdot p_3) = m_\mu^2 E^2$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{g_e^2\hbar}{8\pi p^2\sin^2(\frac{\theta}{2})}\right)^2\left[(m_e c)^2 + \vec{p}^{\,2}\cos^2\left(\frac{\theta}{2}\right)\right]$$

<u>Mott Formula</u>

Note: $g_e = e\sqrt{\frac{4\pi}{\hbar c}} = \sqrt{4\pi\alpha}$

In the non-relativistic limit $\vec{p}^{\,2} \ll (m_e c)^2$ this becomes:

$$\frac{d\sigma}{d\Omega} = \left(\frac{g_e^2\hbar m_e c}{8\pi m_e^2 v^2\sin^2(\frac{\theta}{2})}\right)^2 = \left(\frac{e^2 4\pi\hbar m_e c}{8\pi\hbar c\, m_e^2 v^2 \sin^2(\frac{\theta}{2})}\right)^2 = \left(\frac{e^2}{2m_e v^2\sin^2(\frac{\theta}{2})}\right)^2$$

<u>Rutherford Formula</u>